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The Virial Theorem in Hydromagnetics

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A tensor form of the virial theorem appropriate for configurations in hydromagnetic equilibrium is obtained. Various elementary consequences of this theorem are derived such as the impossibility of spherical symmetry for static configurations with prevailing magnetic fields. A variational form of the virial theorem governing small departures from an initial static state is obtained; and the usefulness of this variational form for estimating the characteristic periods of oscillation of a hydromagnetic system is illustrated by considering a special case.

I. INTRODUCTION

The usual form of the virial theorem was extended to hydromagnetics by Chandrasekhar and Fermi [1]. In this paper we shall consider a generalization of this theorem.

II. THE VIRIAL THEOREM

Consider an inviscid fluid of zero electrical resistivity in which a magnetic field $\mathbf{H}(\mathbf{x})$ prevails. Suppose that the fluid is a perfect gas and that the ratio of the specific heats is γ. Suppose further that apart from the prevailing magnetic field and gas pressure, the only force acting on the medium is that derived from its own gravitation. Under these circumstances the equation of motion governing the fluid velocities is

$$\rho \frac{du_i}{dt} = -\frac{\partial}{\partial x_i}\left(p + \frac{|\mathbf{H}|^2}{8\pi}\right) + \rho \frac{\partial \mathscr{V}}{\partial x_i} + \frac{1}{4\pi}\frac{\partial}{\partial x_j} H_i H_j, \tag{1}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j} \tag{2}$$

is the total time derivative. In Eq. (1), $\mathscr{V}$ denotes the gravitational potential, and the rest of the symbols have their usual meanings. (We are setting $\mu = 1$ in the present analysis.)

A. Some Definitions and Relations

Since we have supposed that the gravitational potential, $\mathscr{V}(\mathbf{x})$, is derived from the distribution of the matter present,

$$\mathscr{V}(\mathbf{x}) = G \int_V \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}' \tag{3}$$

where for the sake of brevity we have written

$$d\mathbf{x}' = dx_1' \, dx_2' \, dx_3' \tag{4}$$

and abridged three integral signs into one. In (3), the integration is effected over the entire volume V occupied by the fluid.

We shall find it convenient to define the symmetric tensor,

$$\mathscr{V}_{ik}(\mathbf{x}) = G \int_V \rho(\mathbf{x})' \frac{(x_i - x_i')(x_j - x_j')}{|\mathbf{x} - \mathbf{x}'|^3} d\mathbf{x}' \tag{5}$$

which represents a generalization of (3) and to which it reduces on contraction:

$$\mathscr{V}_{ii} = \mathscr{V}. \tag{6}$$

Similarly, in generalization of the usual definition of the gravitational potential energy, we shall define

$$\mathscr{W}_{ik} = -\tfrac{1}{2} G \int_V \int_V \rho(\mathbf{x}) \rho(\mathbf{x}') \frac{(x_i - x_i')(x_k - x_k')}{|\mathbf{x} - \mathbf{x}'|^3} d\mathbf{x} \, d\mathbf{x}'; \tag{7}$$

and the contraction of this tensor gives the gravitational potential energy;

$$\mathscr{W}_{ii} = \mathscr{W} = -\tfrac{1}{2} G \int_V \int_V \frac{\rho(\mathbf{x}) \rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x} \, d\mathbf{x}'. \tag{8}$$

An identity which follows from the foregoing definitions is

$$\mathscr{W}_{ik} = \int_V \rho(\mathbf{x}) x_i \frac{\partial \mathscr{V}}{\partial x_k} d\mathbf{x} = \int_V \rho(\mathbf{x}) x_k \frac{\partial \mathscr{V}}{\partial x_i} d\mathbf{x}. \tag{9}$$

This can be established as follows: by definition,

$$\int_V d\mathbf{x}\,\rho(\mathbf{x})x_i\frac{\partial\mathscr{V}}{\partial x_k}=G\int_V d\mathbf{x}\,\rho(\mathbf{x})x_i\frac{\partial}{\partial x_k}\int_V d\mathbf{x}'\,\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}$$

$$=-G\int_V\int_V d\mathbf{x}\,d\mathbf{x}'\,\rho(\mathbf{x})\rho(\mathbf{x}')\frac{x_i(x_k-x_k')}{|\mathbf{x}-\mathbf{x}'|^3}. \tag{10}$$

By transposing the primed and the unprimed variables of integrations in (10) and taking the average of the two equivalent expressions, we verify that the double integral on the last line of (10) is the same as defined in Eq. (7).

In addition to $\mathscr{V}_{ik}$ and $\mathscr{W}_{ik}$, we shall find it useful to define the further tensors

$$\mathscr{T}_{ik}=\tfrac{1}{2}\int_V \rho u_i u_k\,d\mathbf{x}, \tag{11}$$

and[1]

$$\mathscr{M}_{ik}=\frac{1}{8\pi}\int_V H_i H_k\,d\mathbf{x}; \tag{12}$$[1]

and the contraction of these tensors gives the kinetic and the magnetic energies of the system:

$$\mathscr{T}=\mathscr{T}_{ii}=\tfrac{1}{2}\int_V \rho|\mathbf{u}|^2\,d\mathbf{x}, \tag{13}$$

and

$$\mathscr{M}=\mathscr{M}_{ii}=\frac{1}{8\pi}\int_V |\mathbf{H}|^2\,d\mathbf{x}. \tag{14}$$

The analogous expression for the internal heat energy of the system is

$$\mathscr{U}=\frac{1}{\gamma-1}\int_V p\,d\mathbf{x}. \tag{15}$$

[1] The specification of the volume V over which this integration is effected requires some care; we return to this matter in Section I, B below.

Even as $\mathscr{W}_{ik}$, $\mathscr{T}_{ik}$, $\mathscr{M}_{ik}$, and $\mathscr{U}$ characterize, in terms of a few parameters, the distribution of the different forms of energy in the configuration so does the *inertia-tensor*,

$$I_{ik} = \int_V \rho x_i x_k \, d\mathbf{x}, \tag{16}$$

similarly characterize the distribution of the density in the configuration. The contraction of I_{ik} gives the moment of inertia:

$$I = I_{ii} = \int_V \rho |\mathbf{x}|^2 \, d\mathbf{x}. \tag{17}$$

B. The General Form of the Virial Theorem

Returning to Eq. (1), multiply it by x_k and integrate it over the *entire* volume V in which the fluid *and* the field pervade. The left-hand side of the equation can be reduced in the manner:

$$\int_V \rho x_k \frac{du_i}{dt} d\mathbf{x} = \int_V \rho x_k \frac{d^2 x_i}{dt^2} d\mathbf{x} = \int_V \rho \frac{d}{dt}\left(x_k \frac{dx_i}{dt}\right) d\mathbf{x} - \int_V \rho \frac{dx_k}{dt}\frac{dx_i}{dt} d\mathbf{x}$$

$$= \int_V \rho \frac{d}{dt}\left(x_k \frac{dx_i}{dt}\right) d\mathbf{x} - 2\mathscr{T}_{ik}. \tag{18}$$

The terms on the right-hand side, similarly, give

$$-\int_V x_k \frac{\partial}{\partial x_i}\left(p + \frac{|\mathbf{H}|^2}{8\pi}\right) d\mathbf{x} = -\int_S \left(p + \frac{|\mathbf{H}|^2}{8\pi}\right) x_k \, dS_i + \delta_{ik} \int_V \left(p + \frac{|\mathbf{H}|^2}{8\pi}\right) d\mathbf{x}$$

$$= -\int_S \left(p + \frac{|\mathbf{H}|^2}{8\pi}\right) x_k \, dS_i + \delta_{ik} \{(\gamma - 1)\mathscr{U} + \mathscr{M}\}, \tag{19}$$

$$\int_V \rho x_k \frac{\partial \mathscr{V}}{\partial x_i} d\mathbf{x} = \mathscr{W}_{ik}, \tag{20}$$

and

$$\frac{1}{4\pi}\int_V x_k \frac{\partial}{\partial x_j} H_i H_j \, d\mathbf{x} = \frac{1}{4\pi}\int_S x_k H_i H_j \, dS_j - \frac{1}{4\pi}\int_V H_i H_k \, d\mathbf{x}$$

$$= \frac{1}{4\pi}\int_S x_k H_i H_j \, dS_j - 2\mathscr{M}_{ik}, \tag{21}$$

where in Eqs. (19) and (21) the surface integrals are extended over the surface S, bounding V.

Combining Eqs. (18)—(21), we obtain

$$\int_V \rho \frac{d}{dt}\left(x_k \frac{dx_i}{dt}\right) d\mathbf{x} = 2\mathscr{T}_{ik} + \delta_{ik}\{(\gamma - 1)\mathscr{U} + \mathscr{M}\} \tag{22}$$

$$+ \mathscr{W}_{ik} - 2\mathscr{M}_{ik} + \frac{1}{8\pi}\int_S x_k(2H_i H_j\, dS_j - |\mathbf{H}|^2\, dS_i) - \int_S p x_k\, dS_i.$$

We shall suppose that the volume V over which the integrations are extended includes the *whole* system, so that all the variables p, ρ, and $\mathbf{H}$ may be assumed to vanish on S. It is important to note that this definition of V may require us to include in it volumes which one may normally consider as external to the "natural" boundary of the system, namely, the surface on which the density ρ and the material pressure p vanish. The assumption that $\mathbf{H}$ vanishes on S may, indeed, require us to place S at infinity. This latter possibility arises because magnetic fields can extend far beyond the conventional limits of a material object; but to the extent the object is the seat of the magnetic field, there is justification in including all portions of space into which the field extends as parts of the system. And since the field of an object isolated in space must decrease, at least as rapidly as that of a dipole (i.e., as r^{-3}), the surface integrals over the components of $\mathbf{H}$ in Eq. (22) will vanish under these circumstances when $S \to \infty$. However, it may sometimes be convenient to let S coincide with the natural boundary, in which case the surface integrals over S in Eq. (22) must be retained; and, moreover, $\mathscr{M}_{ik}$ will then refer to only that part of the field which is interior to S.

In the present analysis we shall suppose that V includes (as we have already remarked) all parts of space in which the fluid *and* the field pervade. With this explicit understanding, Eq. (22) becomes

$$\int_V \rho \frac{d}{dt}\left(x_k \frac{dx_i}{dt}\right) d\mathbf{x} = 2\mathscr{T}_{ik} - 2\mathscr{M}_{ik} + \mathscr{W}_{ik} + \delta_{ik}\{(\gamma - 1)\mathscr{U} + \mathscr{M}\}. \tag{23}$$

Since all the tensors on the right-hand of this equation are symmetric, the tensor on the left-hand side of the equation must also be symmetric. Therefore,

$$\int_V \rho \frac{d}{dt}\left(x_k \frac{dx_i}{dt}\right) d\mathbf{x} = \int_V \rho \frac{d}{dt}\left(x_i \frac{dx_k}{dt}\right) d\mathbf{x}. \tag{24}$$

An immediate consequence of this result is

$$\int_V \rho \frac{d}{dt}\left(x_k \frac{dx_i}{dt} - x_i \frac{dx_k}{dt}\right) d\mathbf{x} = \frac{d}{dt} \int_V \rho \left(x_k \frac{dx_i}{dt} - x_i \frac{dx_k}{dt}\right) d\mathbf{x}, \tag{25}$$

where in taking d/dt outside of the integral sign, we have made use of the constancy of $\rho\, d\mathbf{x}$ assured by the equation of continuity. *Equation* (25) *expresses the constancy of the total angular momentum of the system.* It is worth noting that the existence of this integral of the equations of motion has not been affected by the presence of the magnetic field.

A further consequence of the identity (24) is that the quantity on the left-hand side of Eq. (23) can be replaced by

$$\frac{1}{2} \int_V \rho \frac{d}{dt}\left(x_k \frac{dx_i}{dt} + x_i \frac{dx_k}{dt}\right) d\mathbf{x} = \frac{1}{2} \frac{d^2}{dt^2} \int_V \rho x_i x_k \, d\mathbf{x} = \frac{1}{2} \frac{d^2 I_{ik}}{dt^2}. \tag{26}$$

With this replacement, Eq. (23) gives

$$\frac{1}{2} \frac{d^2 I_{ik}}{dt^2} = 2\mathscr{T}_{ik} - 2\mathscr{M}_{ik} + \mathscr{W}_{ik} + \delta_{ik}\{(\gamma - 1)\mathscr{U} + \mathscr{M}\}. \tag{27}$$

This equation represents the complete statement of *the virial theorem for hydromagnetics.*

When $i \neq k$, Eq. (27) gives

$$\frac{1}{2} \frac{d^2 I_{ik}}{dt^2} = 2\mathscr{T}_{ik} - 2\mathscr{M}_{ik} + \mathscr{W}_{ik} \qquad (i \neq k); \tag{28}$$

and by contracting the indices in Eq. (27), we obtain

$$\frac{1}{2} \frac{d^2 I}{dt^2} = 2\mathscr{T} + \mathscr{M} + \mathscr{W} + 3(\gamma - 1)\mathscr{U}. \tag{29}$$

C. The Virial Theorem for Equilibrium Configurations

For configurations in equilibrium and in a steady state, Eqs. (27) and (29) give

$$2\mathscr{T}_{ik} - 2\mathscr{M}_{ik} + \mathscr{W}_{ik} + \delta_{ik}\{(\gamma - 1)\mathscr{U} + \mathscr{M}\} = 0, \tag{30}$$

and

$$2\mathscr{T} + \mathscr{M} + \mathscr{W} + 3(\gamma - 1)\mathscr{U} = 0. \tag{31}$$

An alternative way of stating the result expressed by Eq. (30) is that *the tensor,* $2(\mathscr{T}_{ik} - \mathscr{M}_{ik}) + \mathscr{W}_{ik}$, *is isotropic.*

Consider first the case when there are no fluid motions or magnetic fields. In this case, Eq. (30) gives

$$\mathscr{W}_{ik} = \int_V \rho(\mathbf{x}) x_i \frac{\partial \mathscr{V}}{\partial x_k} d\mathbf{x} = -(\gamma - 1)\mathscr{U}\delta_{ik}. \tag{32}$$

This relation is compatible with a spherical symmetry of the underlying density distribution. It would be of interest to know if, *conversely*, the spherical symmetry can be *deduced* from Eq. (32).

When $\mathscr{T}_{ik}$ and $\mathscr{M}_{ik}$ are not zero, $\mathscr{W}_{ik}$ will not, in general, be diagonal. Consequently, *a spherical symmetry of the configuration is, in general, incompatible with the presence of fluid motions and magnetic fields.* An exception is possible if $\mathscr{T}_{ik} \equiv \mathscr{M}_{ik}$ — an identity which is satisfied, for example, by the equipartition solution (2), $u_i = \pm H_i/\sqrt{(4\pi\rho)}$; for, when $\mathscr{T}_{ik} \equiv \mathscr{M}_{ik}$, Eq. (30) gives

$$\mathscr{W}_{ik} = \int_V \rho(\mathbf{x}) x_i \frac{\partial \mathscr{V}}{\partial x_k} d\mathbf{x} = -\{(\gamma - 1)\mathscr{U} + \mathscr{M}\}\delta_{ik}; \tag{33}$$

and this equation is not incompatible with spherical symmetry.

In the special case when only magnetic fields are present,

$$\mathscr{W}_{ik} = 2\mathscr{M}_{ik} - \delta_{ik}\{(\gamma - 1)\mathscr{U} + \mathscr{M}\}; \tag{34}$$

and the impossibility of spherical symmetry, in general, follows from the relation

$$\mathscr{W}_{ik} = 2\mathscr{M}_{ik} \neq 0 \qquad \text{for} \qquad i \neq k. \tag{35}$$

Consider next the contracted form, (31), of the general relation. If $\mathscr{E}$ denotes the total energy of the configuration,

$$\mathscr{E} = \mathscr{T} + \mathscr{U} + \mathscr{M} + \mathscr{W}. \tag{36}$$

From Eqs. (31) and (36) it would appear that as far as these scalar equations go, the magnetic energy can be considered together with the gravitational potential energy. Thus, by eliminating $\mathscr{M} + \mathscr{W}$ between the two equations, we obtain the relation,

$$\mathscr{E} = -\mathscr{T} - (3\gamma - 4)\mathscr{U}, \tag{37}$$

which is independent of $\mathscr{M}$. Alternatively, by eliminating $\mathscr{U}$, we obtain

$$\mathscr{E} = \frac{1}{3(\gamma - 1)}\{(3\gamma - 4)(\mathscr{M} + \mathscr{W}) + (3\gamma - 5)\mathscr{T}\}. \tag{38}$$

In case $\mathscr{T} = 0$, Eqs. (37) and (38) give

$$\mathscr{E} = -(3\gamma - 4)\mathscr{U}, \tag{39}$$

and

$$\mathscr{E} = \frac{3\gamma - 4}{3(\gamma - 1)} (\mathscr{M} + \mathscr{W}). \tag{40}$$

III. The Virial Theorem for Small Oscillations about Equilibrium

We shall now consider small oscillations about equilibrium of a configuration in which magnetic fields are present. We shall, however, suppose that in the stationary state there are no fluid motions; and we shall seek the form which the virial theorem takes under these circumstances.

Considering periodic oscillations with a gyration frequency σ, let $\boldsymbol{\xi} e^{i\sigma t}$ denote the Lagrangian displacement of an element of mass, $dm = \rho d\mathbf{x}$, from its equilibrium position at $\mathbf{x}$. Let $\delta p e^{i\sigma t}$, $\delta \rho e^{i\sigma t}$, and $\delta \mathbf{H} e^{i\sigma t}$ denote the corresponding changes in the other physical variables as we follow the fluid element with its motion. The equation of continuity ensures the constancy of dm for such Lagrangian displacements; and if we suppose that the oscillations take place adiabatically, then

$$\frac{\delta p}{p} = \gamma \frac{\delta \rho}{\rho}, \tag{41}$$

where

$$\frac{\delta \rho}{\rho} = - \operatorname{div} \boldsymbol{\xi}. \tag{42}$$

If $\delta I_{ik} e^{i\sigma t}$, $\delta \mathscr{M}_{ik} e^{i\sigma t}$, etc., denote the first order changes in the various integrals representing these quantities in the stationary state, then Eq. (27) gives

$$- \tfrac{1}{2}\sigma^2 \, \delta I_{ik} = \delta \mathscr{W}_{ik} + \delta_{ik} \{(\gamma - 1)\, \delta \mathscr{U} + \delta \mathscr{M}\} - 2\delta \mathscr{M}_{ik}, \tag{43}$$

the term in $\mathscr{T}_{ik}$ makes no contribution in this order since we have supposed that there are no zero-order fluid motions.

We shall now consider, in turn, the various quantities occurring in Eq. (43). Clearly,

$$\delta I_{ik} = \int_V \rho(\xi_i x_k + x_i \xi_k)\, d\mathbf{x}. \tag{44}$$

Also, according to the definitions of $\mathscr{V}_{ik}$ and $\mathscr{W}_{ik}$ given in Eqs. (5) and (7),

$$\begin{aligned}\delta\mathscr{W}_{ik} &= -G\int_V\int_V dm dm' \xi_j \frac{\partial}{\partial x_j}\frac{(x_i - x_i')(x_k - x_k')}{|\mathbf{x} - \mathbf{x}'|^3} \\ &= -G\int_V d\mathbf{x}\,\rho(\mathbf{x})\xi_j \frac{\partial}{\partial x_j}\int_V d\mathbf{x}'\rho(\mathbf{x}')\frac{(x_i - x_i')(x_k - x_k')}{|\mathbf{x} - \mathbf{x}'|^3} \\ &= -\int_V d\mathbf{x}\,\rho(\mathbf{x})\xi_j \frac{\partial \mathscr{V}_{ik}}{\partial x_j}. \qquad (45)\end{aligned}$$

This last relation is a generalization of the familiar formula,

$$\delta\mathscr{W} = -\int_V d\mathbf{x}\,\rho(\mathbf{x})\xi_j \frac{\partial \mathscr{V}}{\partial x_j}, \qquad (46)$$

for the first order change in the gravitational potential energy consequent to a slight redistribution of the matter in the system.

Turning next to the change in the internal energy, we have

$$(\gamma - 1)\delta\mathscr{U} = \int_V \delta\left(\frac{p}{\rho}\right)\rho\, d\mathbf{x}, \qquad (47)$$

or making use of the relations (41) and (42), we have

$$(\gamma - 1)\,\delta\mathscr{U} = (\gamma - 1)\int \frac{p}{\rho}\frac{\delta\rho}{\rho}\rho\, d\mathbf{x} = -(\gamma - 1)\int_V p\frac{\partial \xi_j}{\partial x_j} d\mathbf{x}. \qquad (48)$$

Letting

$$\Pi = p + \frac{|\mathbf{H}|^2}{8\pi}, \qquad (49)$$

we can write

$$\delta\mathscr{U} = -\int_V \Pi \frac{\partial \xi_j}{\partial x_j} d\mathbf{x} + \frac{1}{8\pi}\int_V |\mathbf{H}|^2 \frac{\partial \xi_j}{\partial x_j} d\mathbf{x}. \qquad (50)$$

Integrating by parts the first of the two integrals on the right-hand side, we obtain

$$\int_V \Pi \frac{\partial \xi_j}{\partial x_j} d\mathbf{x} = \int_S \Pi\, \xi_j\, dS_j - \int_V \xi_j \frac{\partial \Pi}{\partial x_j} d\mathbf{x}. \qquad (51)$$

We shall suppose that S is so placed that all the physical variables and their perturbations vanish on S. (This may require that S be placed at infinity.) Then

$$\int_V \Pi \frac{\partial \xi_j}{\partial x_j} d\mathbf{x} = -\int_V \xi_j \frac{\partial \Pi}{\partial x_j} d\mathbf{x}; \tag{52}$$

or, making use of the relation,

$$\frac{\partial \Pi}{\partial x_j} = \rho \frac{\partial \mathscr{V}}{\partial x_j} + \frac{1}{4\pi} \frac{\partial}{\partial x_l} H_l H_j, \tag{53}$$

which obtains in equilibrium, we have

$$\int_V \Pi \frac{\partial \xi_j}{\partial x_j} d\mathbf{x} = -\int_V \rho \xi_j \frac{\partial \mathscr{V}}{\partial x_j} d\mathbf{x} - \frac{1}{4\pi} \int_V \xi_j \frac{\partial}{\partial x_l} H_l H_j d\mathbf{x}. \tag{54}$$

Integrating by parts the second of the two integrals on the right-hand side of this equation, we obtain

$$\int_V \Pi \frac{\partial \xi_j}{\partial x_j} d\mathbf{x} = -\int_V \rho \xi_j \frac{\partial \mathscr{V}}{\partial x_j} d\mathbf{x} + \frac{1}{4\pi} \int_V H_l H_j \frac{\partial \xi_j}{\partial x_l} d\mathbf{x}, \tag{55}$$

the integrated part, again, making no contribution. Now combining Eqs. (50) and (55), we have

$$\delta \mathscr{U} = \int_V \rho \xi_j \frac{\partial \mathscr{V}}{\partial x_j} d\mathbf{x} + \frac{1}{8\pi} \int_V \left(|\mathbf{H}|^2 \frac{\partial \xi_j}{\partial x_j} - 2 H_l H_j \frac{\partial \xi_l}{\partial x_j} \right) d\mathbf{x}. \tag{56}$$

Considering, finally, $\delta \mathscr{M}_{ik}$, we have

$$\delta \mathscr{M}_{ik} = \frac{1}{8\pi} \int_V (H_k \delta H_i + H_i \delta H_k) d\mathbf{x} + \frac{1}{8\pi} \int_V H_i H_k \frac{\partial \xi_j}{\partial x_j} d\mathbf{x}, \tag{57}$$

where the second term arises from allowing for the variation of the volume element, $d\mathbf{x}$, following the motion in accordance with Eq. (42) and the constancy of $dm = \rho\, d\mathbf{x}$.

Now the change in the magnetic field, $\delta \mathbf{H}$, as we follow the motion of the fluid element, is given by

$$\delta \mathbf{H} = \text{Curl} (\boldsymbol{\xi} \times \mathbf{H}) + (\boldsymbol{\xi} \cdot \text{grad}) \mathbf{H}. \tag{58}$$

The first term on the right-hand side gives the Eulerian change in $\mathbf{H}$ at a *fixed point* caused by the redistribution of the matter, while the second term is the allowance for the fact that, as we follow the fluid element, it finds itself in a slightly displaced location. In the notation of Cartesian tensors, Eq. (58) becomes

$$\delta H_i = H_j \frac{\partial \xi_i}{\partial x_j} - H_i \frac{\partial \xi_j}{\partial x_j}. \tag{59}$$

Making use of this relation, we find that Eq. (57) reduces to

$$\delta \mathscr{M}_{ik} = \frac{1}{8\pi} \int_V \left\{ H_j \left(H_k \frac{\partial \xi_i}{\partial x_j} + H_i \frac{\partial \xi_k}{\partial x_j} \right) - H_i H_k \frac{\partial \xi_j}{\partial x_j} \right\} d\mathbf{x}; \tag{60}$$

and by contracting, we find

$$\delta \mathscr{M} = -\frac{1}{8\pi} \int_V \left(|\mathbf{H}|^2 \frac{\partial \xi_j}{\partial x_j} - 2 H_j H_l \frac{\partial \xi_l}{\partial x_j} \right) d\mathbf{x}. \tag{61}$$

Now inserting in Eq. (43) for δI_{ik}, $\delta \mathscr{W}_{ik}$, etc., the expressions we have derived for them, we find:

$$\begin{aligned} -\tfrac{1}{2}\sigma^2 \int_V \rho (\xi_i x_k + x_i \xi_k)\, d\mathbf{x} &= -\int_V \rho \xi_j \frac{\partial \mathscr{V}_{ik}}{\partial x_j} d\mathbf{x} \\ &+ \delta_{ik} \left\{ (\gamma - 1) \int_V \rho \xi_j \frac{\partial \mathscr{V}}{\partial x_j} d\mathbf{x} + \frac{\gamma - 2}{8\pi} \int_V \left(|\mathbf{H}|^2 \frac{\partial \xi_j}{\partial x_j} - 2 H_j H_l \frac{\partial \xi_l}{\partial x_j} \right) d\mathbf{x} \right. \\ &+ \frac{1}{4\pi} \int_V \left\{ H_i H_k \frac{\partial \xi_j}{\partial x_j} - H_j \left(H_k \frac{\partial \xi_i}{\partial x_j} + H_i \frac{\partial \xi_k}{\partial x_j} \right) \right\} d\mathbf{x}. \end{aligned} \tag{62}$$

By contracting this equation, we obtain

$$\begin{aligned} -\sigma^2 \int_V \rho \xi_i x_i\, d\mathbf{x} & \\ = (3\gamma - 4) \left\{ \int_V \rho \xi_j \frac{\partial \mathscr{V}}{\partial x_j} d\mathbf{x} \right. &+ \frac{1}{8\pi} \int_V \left(|\mathbf{H}|^2 \frac{\partial \xi_j}{\partial x_j} - 2 H_j H_l \frac{\partial \xi_l}{\partial x_j} \right) d\mathbf{x}. \end{aligned} \tag{63}$$

A. A Characteristic Equation for Determining the Periods of Oscillation

Equations (62) and (63) can be used to obtain estimates for σ^2 by inserting in them suitable "trial" functions for $\boldsymbol{\xi}$. Thus, the simplest assumption,

$$\boldsymbol{\xi} = \text{constant } \mathbf{x} \tag{64}$$

together with Eq. (63), yields the formula (3)

$$\sigma^2 = -(3\gamma - 4)\frac{\mathscr{W} + \mathscr{M}}{I}. \tag{65}$$

However, since a configuration with a prevalent magnetic field cannot be spherically symmetric, the use of a trial function which corresponds to a uniform radial expansion would appear unsuitable; indeed, the substitution of (64) in the tensor equation (62) will lead to gross inconsistencies unless the departures from spherical symmetry of the equilibrium configuration are small. If the latter should not be the case, a more consistent procedure would appear to be the following.

Assume that

$$\boldsymbol{\xi} = \boldsymbol{X}\mathbf{x} \tag{66}$$

where $\boldsymbol{X}$ is a symmetric matrix:

$$\mathscr{X}_{ij} = \mathscr{X}_{ji}. \tag{67}$$

Further, define the super-matrix

$$\mathscr{W}_{lj;ik} = \int_V \rho(\mathbf{x})\, x_l \frac{\partial}{\partial x_j} \mathscr{V}_{ik}\, d\mathbf{x}. \tag{68}$$

By contracting this super-matrix with respect to i and k, we obtain

$$\mathscr{W}_{lj;ii} = \int_V \rho(\mathbf{x}) x_l \frac{\partial \mathscr{V}}{\partial x_j} d\mathbf{x} = \mathscr{W}_{lj}. \tag{69}$$

Also, let

$$\boldsymbol{M} = (\mathscr{M}_{ik}), \qquad \boldsymbol{W} = (W_{ik}) \qquad \text{and} \qquad \boldsymbol{I} = (I_{ik}). \tag{70}$$

Clearly,

$$\mathscr{M} = \mathscr{M}_{ii} = \mathrm{Tr}\,(\boldsymbol{M}) \qquad \text{and} \qquad \mathscr{W} = \mathscr{W}_{ii} = \mathrm{Tr}\,(\boldsymbol{W}), \tag{71}$$

where Tr stands for trace (meaning diagonal sum). Similarly,

$$\frac{\partial \xi_j}{\partial x_j} = \mathrm{Tr}\,(\boldsymbol{X}). \tag{72}$$

With the foregoing definitions, the substitution of the trial function (66) in Eq. (62) leads to the result

$$\begin{aligned} -\tfrac{1}{2}\sigma^2 (\mathbf{I}\boldsymbol{X} + \boldsymbol{X}\mathbf{I})_{ik} = {} & -\mathscr{X}_{jl}\mathscr{W}_{lj;ik} \\ & + \delta_{ik}\{(\gamma - 1)\,\mathrm{Tr}\,(\boldsymbol{XW}) + (\gamma - 2)\,[\mathrm{Tr}\,(\boldsymbol{X})\,\mathrm{Tr}\,(\boldsymbol{M}) - 2\,\mathrm{Tr}\,(\boldsymbol{XM})]\} \\ & + 2\,\{\mathrm{Tr}\,(\boldsymbol{X})\mathscr{M}_{ik} - [\boldsymbol{XM} + \boldsymbol{MX}]_{ik}\}. \end{aligned} \tag{73}$$

Equation (73) represents a system of linear homogeneous equations for the six coefficients of the (assumed) symmetric linear transformation, $\boldsymbol{X}$. The determinant of the system must, therefore, vanish; and the resulting characteristic equation will not only determine σ^2 but also the transformation, $\boldsymbol{X}$ (apart from a constant of proportionality).

Finally, we may note that by contracting the tensor equation (73), we obtain the scalar equation,

$$-\sigma^2 \operatorname{Tr}(\mathbf{I}\boldsymbol{X}) = (3\gamma - 4)\{\operatorname{Tr}(\boldsymbol{X}\boldsymbol{W}) + \operatorname{Tr}(\boldsymbol{X}) \operatorname{Tr}(\boldsymbol{M}) - 2 \operatorname{Tr}(\boldsymbol{X}\boldsymbol{M})\}. \qquad (74)$$

which is a generalization of (65).

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